Recent progress in the theory of effective Kan fibrations in simplicial sets

Benno van den Berg
Institute for Logic, Language and Computation
University of Amsterdam

TYPES 2024
Copenhagen, 11 June 2024

Types as ∞-groupoids

Homotopy type theory starts from a novel interpretation of a type as a *(the homotopy type of) a space* or an ∞-*groupoid*.

Types as ∞-groupoids

Homotopy type theory starts from a novel interpretation of a type as a *(the homotopy type of) a space* or an ∞-*groupoid*.

Why do we want this? It justifies *extensionality principles* like:

1. function extensionality.
2. univalence.
3. propositional truncation.
4. higher-inductive types.

An interpretation of type theory in which we interpret types as ∞-groupoids should validate these principles.

Types as ∞-groupoids

Homotopy type theory starts from a novel interpretation of a type as a *(the homotopy type of) a space* or an ∞-*groupoid*.

Why do we want this? It justifies *extensionality principles* like:

1. function extensionality.
2. univalence.
3. propositional truncation.
4. higher-inductive types.

An interpretation of type theory in which we interpret types as ∞-groupoids should validate these principles.

Question

Why is that so? And is this also true *constructively*?

What is an ∞-groupoid?

There are many possible definitions of an ∞-groupoid; but the most popular one (among mathematicians) is that of a *Kan complex*.

What is an ∞-groupoid?

There are many possible definitions of an ∞-groupoid; but the most popular one (among mathematicians) is that of a *Kan complex*.

Definition (Kan fibration)

A map $f : Y \to X$ of simplicial sets is a *Kan fibration* if every solid commutative square

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & Y \\ \big\downarrow & \nearrow & \big\downarrow f \\ \Delta^n & \longrightarrow & X \end{array}$$

has a (not necessarily unique) dotted filler as shown. A simplicial set Y is a *Kan complex* if $Y \to 1$ is a Kan fibration.

What is an ∞-groupoid?

There are many possible definitions of an ∞-groupoid; but the most popular one (among mathematicians) is that of a *Kan complex*.

Definition (Kan fibration)

A map $f : Y \to X$ of simplicial sets is a *Kan fibration* if every solid commutative square

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & Y \\ \downarrow & \nearrow & \downarrow f \\ \Delta^n & \longrightarrow & X \end{array}$$

has a (not necessarily unique) dotted filler as shown. A simplicial set Y is a *Kan complex* if $Y \to 1$ is a Kan fibration.

In classical maths being a Kan fibration is understood as a *property*.

What is an ∞-groupoid?

There are many possible definitions of an ∞-groupoid; but the most popular one (among mathematicians) is that of a *Kan complex*.

Definition (Kan fibration)

A map $f : Y \to X$ of simplicial sets is a *Kan fibration* if every solid commutative square

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & Y \\ \downarrow & \nearrow & \downarrow f \\ \Delta^n & \longrightarrow & X \end{array}$$

has a (not necessarily unique) dotted filler as shown. A simplicial set Y is a *Kan complex* if $Y \to 1$ is a Kan fibration.

In classical maths being a Kan fibration is understood as a *property*. However, let us say that a map $f : Y \to X$ is a *functional Kan fibration* if it comes equipped with an explicit choice of lifts for any commutative square as the one above.

The simplicial sets model

Theorem (Voevodsky)

The category of simplicial sets carries a model of type theory in which Kan fibrations interpret dependent types. In this model function extensionality and univalence hold.

Theorem (Kan-Quillen)

The category of simplicial sets carries a model structure.

How constructive is this?

The simplicial sets model

Theorem (Voevodsky)

The category of simplicial sets carries a model of type theory in which Kan fibrations interpret dependent types. In this model function extensionality and univalence hold.

Theorem (Kan-Quillen)

The category of simplicial sets carries a model structure.

How constructive is this?

Simon Henry has given a constructive proof of the existence of the Kan-Quillen model structure. This *almost* leads to a constructive model of HoTT (see work with Gambino, Sattler and Szumiło).

The simplicial sets model

Theorem (Voevodsky)

The category of simplicial sets carries a model of type theory in which Kan fibrations interpret dependent types. In this model function extensionality and univalence hold.

Theorem (Kan-Quillen)

The category of simplicial sets carries a model structure.

How constructive is this?

Simon Henry has given a constructive proof of the existence of the Kan-Quillen model structure. This *almost* leads to a constructive model of HoTT (see work with Gambino, Sattler and Szumiło).

Two issues remain:

1. Π-types
2. Coherence issues

Uniformity to the rescue?

Can we do better by imposing uniformity conditions as in the work on cubical type theory?

Uniformity to the rescue?

Can we do better by imposing uniformity conditions as in the work on cubical type theory? Work by Gambino and Sattler on uniform Kan fibrations in simplicial sets suggests the answer could be yes!

Uniformity to the rescue?

Can we do better by imposing uniformity conditions as in the work on cubical type theory? Work by Gambino and Sattler on uniform Kan fibrations in simplicial sets suggests the answer could be yes!

In this talk I will discuss another notion of a uniform Kan fibration: that of an *effective Kan fibration*. This notion was introduced in a book written together with Eric Faber.

Uniformity to the rescue?

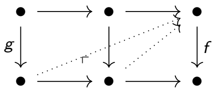
Can we do better by imposing uniformity conditions as in the work on cubical type theory? Work by Gambino and Sattler on uniform Kan fibrations in simplicial sets suggests the answer could be yes!

In this talk I will discuss another notion of a uniform Kan fibration: that of an *effective Kan fibration*. This notion was introduced in a book written together with Eric Faber.

The contents of this talk are mostly based on the preprint *Examples and cofibrant generation of effective Kan fibration in simplicial sets*, arXiv2402.10568, written together with Freek Geerligs.

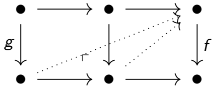
Maps functional Kan fibrations lift against

- If f lifts against g, then f also lifts against any pushout of g.

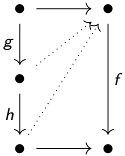


Maps functional Kan fibrations lift against

- If f lifts against g, then f also lifts against any pushout of g.

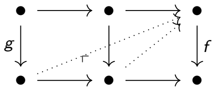


- If f lifts against g and h, then f also lifts against $h \circ g$.

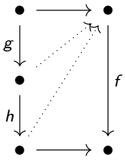


Maps functional Kan fibrations lift against

- If f lifts against g, then f also lifts against any pushout of g.



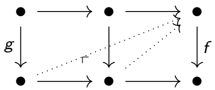
- If f lifts against g and h, then f also lifts against $h \circ g$.



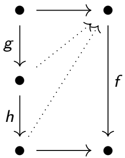
A sequence $S_0 \subseteq S_1 \subseteq \ldots \subseteq S_k$ of subobjects of Δ^n where each $S_i \subseteq S_{i+1}$ is a pushout of a horn inclusion will be called a *horn pushout sequence*.

Maps functional Kan fibrations lift against

- If f lifts against g, then f also lifts against any pushout of g.



- If f lifts against g and h, then f also lifts against $h \circ g$.



A sequence $S_0 \subseteq S_1 \subseteq \ldots \subseteq S_k$ of subobjects of Δ^n where each $S_i \subseteq S_{i+1}$ is a pushout of a horn inclusion will be called a *horn pushout sequence*.
We conclude: *each functional Kan fibration has induced lifts against inclusions $S \subseteq T$ between subobjects of Δ^n if S and T are given as the endpoints of a horn pushout sequence.*

Effective Kan fibrations

What happens if we pull back a horn inclusion along a degeneracy?

$$\begin{array}{ccc} S & \longrightarrow & \Lambda^n_k \\ \downarrow & \lrcorner & \downarrow \\ \Delta^{n+1} & \xrightarrow{s_i} & \Delta^n \end{array}$$

Effective Kan fibrations

What happens if we pull back a horn inclusion along a degeneracy?

$$\begin{array}{ccc} S & \longrightarrow & \Lambda^n_k \\ \big\downarrow & \lrcorner & \big\downarrow \\ \Delta^{n+1} & \xrightarrow{s_i} & \Delta^n \end{array}$$

The inclusion $S \subseteq \Delta^{n+1}$ can be written as the composition of a horn pushout sequence. This decomposition is not unique; however, the induced lift against any functional Kan fibration will be.

Effective Kan fibrations

What happens if we pull back a horn inclusion along a degeneracy?

$$\begin{tikzcd} S \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Lambda^n_k \arrow[d] \\ \Delta^{n+1} \arrow[r, "s_i"] & \Delta^n \end{tikzcd}$$

The inclusion $S \subseteq \Delta^{n+1}$ can be written as the composition of a horn pushout sequence. This decomposition is not unique; however, the induced lift against any functional Kan fibration will be.

Definition (Effective Kan fibration)

A functional Kan fibration $f : Y \to X$ is an *effective Kan fibration* if its induced lifts make any diagram of the following form commute:

$$\begin{tikzcd} S \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Lambda^n_k \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ \Delta^{n+1} \arrow[r, "s_i"] \arrow[urr, dotted] & \Delta^n \arrow[r] \arrow[ur, dotted] & X \end{tikzcd}$$

Properties of effective Kan fibrations

We have established the following properties of effective Kan fibrations:

Classical correctness: Using classical logic and choice, one can show that every Kan fibration can be equipped with the structure of an effective Kan fibration (jww Eric Faber).

Properties of effective Kan fibrations

We have established the following properties of effective Kan fibrations:

Classical correctness: Using classical logic and choice, one can show that every Kan fibration can be equipped with the structure of an effective Kan fibration (jww Eric Faber).

Exponentials: If A is an effective Kan complex, then so is A^B for any simplicial set B. More generally, effective Kan fibrations are closed under Π (jww Eric Faber).

Properties of effective Kan fibrations

We have established the following properties of effective Kan fibrations:

Classical correctness: Using classical logic and choice, one can show that every Kan fibration can be equipped with the structure of an effective Kan fibration (jww Eric Faber).

Exponentials: If A is an effective Kan complex, then so is A^B for any simplicial set B. More generally, effective Kan fibrations are closed under Π (jww Eric Faber).

Other type constructors: Effective Kan fibrations interpret the following type constructors: $\Pi, \Sigma, +, \times, 0, 1, \mathbb{N}$. We have a slightly ineffective proof for W as well (jww Shinichiro Tanaka).

Properties of effective Kan fibrations

We have established the following properties of effective Kan fibrations:

Classical correctness: Using classical logic and choice, one can show that every Kan fibration can be equipped with the structure of an effective Kan fibration (jww Eric Faber).

Exponentials: If A is an effective Kan complex, then so is A^B for any simplicial set B. More generally, effective Kan fibrations are closed under Π (jww Eric Faber).

Other type constructors: Effective Kan fibrations interpret the following type constructors: $\Pi, \Sigma, +, \times, 0, 1, \mathbb{N}$. We have a slightly ineffective proof for W as well (jww Shinichiro Tanaka).

Other examples: Simplicial groups (more generally, simplicial Malcev algebras) are Kan (jww Freek Geerligs).

Properties of effective Kan fibrations

We have established the following properties of effective Kan fibrations:

Classical correctness: Using classical logic and choice, one can show that every Kan fibration can be equipped with the structure of an effective Kan fibration (jww Eric Faber).

Exponentials: If A is an effective Kan complex, then so is A^B for any simplicial set B. More generally, effective Kan fibrations are closed under Π (jww Eric Faber).

Other type constructors: Effective Kan fibrations interpret the following type constructors: $\Pi, \Sigma, +, \times, 0, 1, \mathbb{N}$. We have a slightly ineffective proof for W as well (jww Shinichiro Tanaka).

Other examples: Simplicial groups (more generally, simplicial Malcev algebras) are Kan (jww Freek Geerligs).

A big open problem

We can construct universes using the Hofmann-Streicher construction.

Properties of effective Kan fibrations

We have established the following properties of effective Kan fibrations:

Classical correctness: Using classical logic and choice, one can show that every Kan fibration can be equipped with the structure of an effective Kan fibration (jww Eric Faber).

Exponentials: If A is an effective Kan complex, then so is A^B for any simplicial set B. More generally, effective Kan fibrations are closed under Π (jww Eric Faber).

Other type constructors: Effective Kan fibrations interpret the following type constructors: $\Pi, \Sigma, +, \times, 0, 1, \mathbb{N}$. We have a slightly ineffective proof for W as well (jww Shinichiro Tanaka).

Other examples: Simplicial groups (more generally, simplicial Malcev algebras) are Kan (jww Freek Geerligs).

A big open problem

We can construct universes using the Hofmann-Streicher construction. However, I do not know if they are effectively Kan or satisfy univalence.

Algebraic weak factorisation systems

Theorem (Freek Geerligs & BvdB)

The effective Kan fibrations are cofibrantly generated by a small double category.

Theorem (Bourke & Garner)

If a class of maps in a presheaf category is cofibrantly generated by a small double category, then it is the class of right maps in an algebraic weak factorisation system (AWFS).

Algebraic weak factorisation systems

Theorem (Freek Geerligs & BvdB)

The effective Kan fibrations are cofibrantly generated by a small double category.

Theorem (Bourke & Garner)

If a class of maps in a presheaf category is cofibrantly generated by a small double category, then it is the class of right maps in an algebraic weak factorisation system (AWFS).

The proof given by Bourke and Garner is classical; however, I am convinced that it can be made constructive in the case at hand (joint work in progress with Paul Seip and John Bourke).

Algebraic weak factorisation systems

Theorem (Freek Geerligs & BvdB)

The effective Kan fibrations are cofibrantly generated by a small double category.

Theorem (Bourke & Garner)

If a class of maps in a presheaf category is cofibrantly generated by a small double category, then it is the class of right maps in an algebraic weak factorisation system (AWFS).

The proof given by Bourke and Garner is classical; however, I am convinced that it can be made constructive in the case at hand (joint work in progress with Paul Seip and John Bourke).

Ultimately I hope the effective Kan fibrations can be the fibrations in an algebraic model structure and the dependent types in a model of homotopy type theory (constructively!).

Extra slides

Double categories

The definition of an effective Kan fibration can be phrased in the language of *double categories*.

Definition (double category)

A double category consists of:

- Objects.
- Horizontal arrows $\bullet \longrightarrow \bullet$ between these objects.
- Vertical arrows $\begin{array}{c}\bullet\\ \downarrow\\ \bullet\end{array}$ between these objects.
- Squares $\begin{array}{ccc}\bullet & \longrightarrow & \bullet\\ \downarrow & & \downarrow\\ \bullet & \longrightarrow & \bullet\end{array}$ which can be composed horizontally and vertically.

Double categories

The definition of an effective Kan fibration can be phrased in the language of *double categories*.

Definition (double category)

A double category consists of:

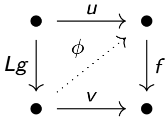
- Objects.
- Horizontal arrows $\bullet \longrightarrow \bullet$ between these objects.
- Vertical arrows $\begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array}$ between these objects.
- Squares $\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$ which can be composed horizontally and vertically.

Example

If $\mathcal{C}$ is a category, then there is a double category $\mathrm{Sq}(\mathcal{C})$ whose horizontal and vertical arrows are the morphisms of $\mathcal{C}$, while its squares are the commutative squares in $\mathcal{C}$.

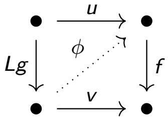
Lifting against double categories

Let $\mathbb{L}$ be a double category and $L : \mathbb{L} \to \mathrm{Sq}(\mathcal{C})$ be a double functor. If $f : Y \to X$ is a morphism in $\mathcal{C}$, then a *right lifting structure against* L is a function which assigns to each vertical morphism g in $\mathbb{L}$ and each square $(u, v) : Lg \to f$ in $\mathrm{Sq}(\mathcal{C})$ a lift $\phi = \phi_g(u, v)$ as shown:

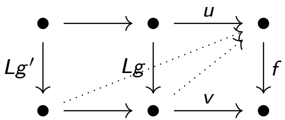


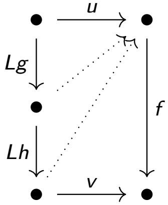
Lifting against double categories

Let $\mathbb{L}$ be a double category and $L : \mathbb{L} \to \mathrm{Sq}(\mathcal{C})$ be a double functor. If $f : Y \to X$ is a morphism in $\mathcal{C}$, then a *right lifting structure against L* is a function which assigns to each vertical morphism g in $\mathbb{L}$ and each square $(u, v) : Lg \to f$ in $\mathrm{Sq}(\mathcal{C})$ a lift $\phi = \phi_g(u, v)$ as shown:



These lifts are required to satisfy two compatibility conditions, a *horizontal* and a *vertical* one, which can be depicted as follows:





A double category for effective Kan fibrations

Let $\mathbb{L}_0$ be the following double category:

- Objects are cofibrant sieves $S \subseteq \Delta^n$.
- Horizontal morphisms are pullback squares $\begin{array}{ccc} S & \longrightarrow & T \\ \downarrow & & \downarrow \\ \Delta^n & \xrightarrow{\alpha} & \Delta^m \end{array}$.
- Vertical morphisms are horn pushout sequences $S_0 \subseteq S_1 \subseteq \ldots \subseteq S_k$.
- A square from $S_0 \subseteq S_1 \subseteq \ldots \subseteq S_k \subseteq \Delta^n$ to $T_0 \subseteq T_1 \subseteq \ldots \subseteq T_l \subseteq \Delta^m$ is given by a map $\alpha : \Delta^n \to \Delta^m$ and a monotone function $f : \{0, \ldots, l\} \to \{0, \ldots, m\}$ such that $f(0) = 0, f(l) = k$ and $\alpha^* T_i = S_{f(i)}$. Such a square is a *face* or *degeneracy square* if α is a face or degeneracy map.

A double category for effective Kan fibrations

Let $\mathbb{L}_0$ be the following double category:

- Objects are cofibrant sieves $S \subseteq \Delta^n$.
- Horizontal morphisms are pullback squares $\begin{array}{ccc} S & \longrightarrow & T \\ \downarrow & & \downarrow \\ \Delta^n & \xrightarrow{\alpha} & \Delta^m \end{array}$.
- Vertical morphisms are horn pushout sequences $S_0 \subseteq S_1 \subseteq \ldots \subseteq S_k$.
- A square from $S_0 \subseteq S_1 \subseteq \ldots \subseteq S_k \subseteq \Delta^n$ to $T_0 \subseteq T_1 \subseteq \ldots \subseteq T_l \subseteq \Delta^m$ is given by a map $\alpha : \Delta^n \to \Delta^m$ and a monotone function $f : \{0, \ldots, l\} \to \{0, \ldots, m\}$ such that $f(0) = 0, f(l) = k$ and $\alpha^* T_i = S_{f(i)}$. Such a square is a *face* or *degeneracy square* if α is a face or degeneracy map.

The double category $\mathbb{L}$ is defined in the same way, but each square is an explicit composition of face and degeneracy squares.

Theorem (Freek Geerligs & BvdB)

A map is an effective Kan fibration iff it has a right lifting structure against the double category $\mathbb{L}$.